Int. J. Heat Mass Transfer. Vol. 22, pp. 99–109
Pergamon Press Ltd. 1979. Printed in Great Britain

THE NON-ADIABATIC CALORIMETER PROBLEM AND ITS APPLICATION TO TRANSFER PROCESSES IN SUSPENSIONS OF SOLIDS

J. W. HEMMINGS*
Chemical Engineering Research Group, Council for Scientific and Industrial Research, P.O. Box 395, Pretoria 0001, Republic of South Africa

and

J. KERN*
Balcke-Dürr A.G., 4030 Ratingen, Federal Republic of Germany

(*Received* 25 *November* 1977)

Abstract—The problem of simultaneous transfer of heat from a fluid of finite extent to the ambient and to a solid of finite thermal conductivity is solved. The general solution is in the form of an infinite series with the Biot number, the thermal extent ratio H, the ambient loss parameter E and the ambient temperature V_a as parameters; the series shows strong exponential decay so that a few terms suffice for most applications. The solution is an extension and refinement of the classical calorimeter problem. The results find application in the analysis of (and design of equipment for) continuous exchange involving suspended solids. The form of the solution is sufficiently simple for optimisation of insulation thickness, residence time and other process parameters. For limiting values of the parameters simplified solutions are presented, for instance for the double pipe heat exchanger with ambient loss.

NOMENCLATURE

A, heat-transfer area between ambient and fluid [m^2];
C, heat capacity [J kg^{-1} K^{-1}];
G, mass rate [kg s^{-1}];
h, heat-transfer coefficient between solid and fluid [Wm^{-2} K^{-1}];
l, position in contactor [m];
L, length of contactor [m];
k, thermal conductivity of solid [Wm^{-1} K^{-1}];
M, mass [kg];
r, radial position within sphere [m];
R, radius of sphere [m];
t, contact time of solid and fluid [s];
S, flow area of contactor [m^2];
T, temperature [K];
U, heat-transfer coefficient between fluid and ambient [Wm^{-2} K^{-1}];
α, thermal diffusivity of solid [m^2 s^{-1}];
β; Δ, defined in text.

Dimensionless quantities

Bi, Biot number between solid and fluid;
E, ambient loss parameter;
E^*, ambient loss parameter used when $k = \infty$;
f, component of dimensionless temperature [defined in equation (32)];
g, component of dimensionless temperature [defined in equation (32)];
H, thermal extent ratio (in batch case), which is ratio of the enthalpy rates in the continuous case;
Kn, n^{th} coefficient in dimensionless temperature, equation (13);
s, Laplace variable corresponding to τ;
V, dimensionless temperature;
x, dimensionless radial variable within the sphere;
y, variable of the transcendental equation (15);
y_n, n^{th} root of the transcendental equation, a parameter of the solution.

Greek symbols

ε, error in energy balance solution, a function of τ;
τ, Fourier number;
ϕ, the LHS of equation (15);
ψ, the RHS of equation (15).

Subscripts

a, ambient;
c, centre of solid;
f, fluid;
fi, fluid inlet (counter-current case);
$f0$, initial fluid;
fe, fluid as predicted by the energy balance solution;
s, solid;
$s0$, initial solid;

*A substantial part of this work was done while the authors were with the Department of Chemical Engineering, University of the Witwatersrand, Johannesburg, 2001 Republic of South Africa.

se, solid as predicted by the energy balance solution;
σ, surfaces.

Superscripts

ˆ, integrated average defined as:

$$\hat{Q} = \frac{3}{R^3}\int_0^R r^2 Q\,\mathrm{d}r, \text{ for any quantity } Q;$$

¯, Laplace transform of a quantity.

1. INTRODUCTION

THE NON-ADIABATIC calorimeter problem is the logical extension of the adiabatic calorimeter problem [1–3].

It can be transformed to the following granular solid–fluid heat- or mass-transfer operations: (a) co-current (or countercurrent) exchange in a moving bed contactor with heat loss to ambient; (b) exchange between a batch of solids and a well mixed fluid stream in a fluidised bed.

This paper extends previous analyses that exist for the adiabatic case [4–6] and supplements the analysis of Siegmund, Munro and Amundson [7] for the non-adiabatic case.

Siegmund *et al.* assume that the mechanism of heat transfer to the ambient is essentially conduction. This assumption is valid in fixed beds where the fluid flow is such that one may assume eddy conductivity to be constant across any cross-section; it is not a useful assumption in dilute fluidised beds and similar systems. Here one can assume that the resistance to heat exchange with the ambient is limited to the wall region; this resistance comprises a forced convection term on the inside and resistance due to insulation and free convection on the outside. We simplify and assume that the heat loss to the ambient is governed by an overall heat-transfer coefficient that does not vary along the contactor.

2. FORMULATION OF THE PROBLEM

We examine the cooling (or heating) of a sphere of uniform initial temperature by a moderately stirred fluid contained in a bath which is not well insulated and therefore capable of losing heat to the ambient. By moderately stirred fluid we mean a fluid that is sufficiently well stirred that it has a bulk temperature but that heat transfer from the fluid bulk to the solid surface is impeded by a film resistance.

This differs from a perfectly stirred fluid where the bulk fluid and solid surface temperature are the same.

We assume further that: (a) the film coefficient between the solid surface and the fluid is constant; (b) the overall heat-transfer coefficient between the fluid and the surroundings is constant.

We feel that the above model is an optimum trade-off between a desire to represent the real situation accurately and the necessity to have a tractable problem. We do not consider, for example, spatial variation of the heat-transfer coefficient due to natural convection. The differential equations governing the rate of heat transfer are the equation for heat conduction, a heat balance at the solid surface and the heat balance for the whole system:

$$\frac{\partial T_s}{\partial t} = \frac{\alpha}{r}\frac{\partial^2 r T_s}{\partial r^2} \quad \text{for } 0 \leqslant t \leqslant \infty,\ 0 \leqslant r \leqslant R, \tag{1}$$

$$-k\left.\frac{\partial T_s}{\partial r}\right|_{r=R} = h[T_s(R,t) - T_f(t)] \tag{2}$$

$$-M_f C_f \frac{\mathrm{d}T_f}{\mathrm{d}t} = M_s C_s \frac{\mathrm{d}\hat{T}_s}{\mathrm{d}t} + UA(T_f - T_a) \tag{3}$$

where

$$\hat{T}_s = \frac{3}{R^3}\int_0^R r^2 T_s(r,t)\,\mathrm{d}r.$$

We have the following boundary and initial conditions to complete the system of equations

$$T_s(r,0) = T_{s0}, \tag{4}$$

$$\left.\frac{\partial T_s}{\partial r}\right|_{r=0} = 0 \tag{5}$$

$$T_f(0) = T_{f0}. \tag{6}$$

We introduce the following dimensionless variables:

$$V_s = (T_s - T_{s0})/(T_{f0} - T_{s0}),$$
$$V_f = (T_f - T_{s0})/(T_{f0} - T_{s0}),$$
$$x = r/R,$$
$$\tau = \alpha t/R^2,$$

and the following parameters

$V_a = (T_a - T_{s0})/(T_{f0} - T_{s0})$, the asymptote ($t \to \infty$) of V_s and V_f,

$Bi = hR/k$, the Biot number,

$H = M_s C_s/M_f C_f$, the thermal extent ratio, and

$E = UAR^2/\alpha M_f C_f$, the ambient loss parameter analogous to a Biot number.

The set of equations (1)–(6) becomes

$$\frac{\partial x V_s}{\partial \tau} = \frac{\partial^2 x V_s}{\partial x^2} \quad \text{for } 0 \leqslant \tau \leqslant \infty,\ 0 \leqslant x \leqslant 1 \tag{7}$$

$$\left.\frac{\partial V_s}{\partial x}\right|_{x=1} = Bi(V_f - V_s|_{x=1}) \tag{8}$$

$$-\frac{\mathrm{d}V_f}{\mathrm{d}\tau} = H\frac{\mathrm{d}\hat{V}_s}{\mathrm{d}\tau} + E(V_f - V_a) \tag{9}$$

$$V_s(x,0) = 0 \tag{10}$$

$$\left.\frac{\partial V_s}{\partial x}\right|_{x=0} = 0 \tag{11}$$

$$V_f(0) = 1. \tag{12}$$

The solution to the above set of equations is readily found by applying Laplace transformation (see Appendix).

3. PRESENTATION OF THE SOLUTION

The solution is:

$$V_s = V_a - 2\sum_{n=1}^{\infty}(EV_a - y_n^2)Kn\frac{\sin(y_n x)}{x\sin y_n}\exp(-y_n^2\tau). \quad (13)$$

The coefficients Kn are given by:

$$Kn = \frac{Bi(E+3BiH-y_n^2)}{y_n^6+[Bi(Bi-1)-6BiH-2E]y_n^4+[9Bi^2H(1+H)+(6BiH-2Bi^2+2Bi+E)E]y_n^2+Bi(Bi-1)E^2-3Bi^2HE} \quad (14)$$

The y_n are the solutions of the equation

$$\frac{\tan y}{y} = \frac{3BiH+E-y^2}{(Bi-1)(y^2-E)+3BiH}. \quad (15)$$

For convenience equation (13) has been written in a form that incorporates $y_0 = 0$ separately and takes account of the fact that $y_n = -y_{-n}$ by multiplying the series by 2. Thus we have to consider only those solutions of (15) that lie in the half-domain Real $\{y\} \geq 0$.

surface:

$$V_\sigma = V_a - 2\sum_{n=1}^{\infty}(EV_a - y_n^2)Kn\exp(-y_n^2\tau) \quad (17)$$

solid average:

$$\hat{V}_s = V_a + 6\sum_{n=1}^{\infty}(EV_a - y_n^2)Kn \times \frac{(y_n\cos y_n - \sin y_n)}{y_n^2\sin y_n}\exp(-y_n^2\tau) \quad (18)$$

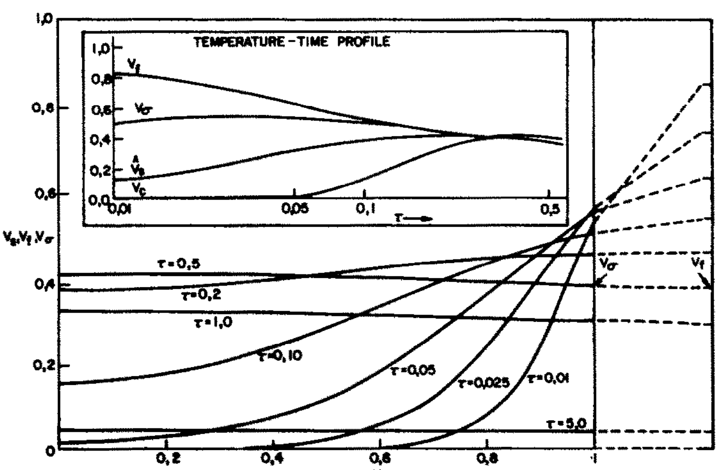

FIG. 1. Temperature–distance profiles within the solid for $Bi = 10$; $H = 1$; $E = 1$ and $V_a = 0$ at various dimensionless times. With temperature time profile for fluid, surface, average and centre temperatures inset.

Figure 1 shows the variation of V_s with x for several values of τ and V_f, V_σ, $\hat{V}_s$, V_c against τ for $Bi = 10, H = 1, E = 1$.

Local, fluid and average temperatures

Often we are not interested in the temperature distribution throughout the solid but only in the minimum, maximum and average solid temperatures. The surface, centre and average temperatures are of interest, as well as the fluid temperatures. These temperatures are given by

centre:

$$V_c = V_a - 2\sum_{n=1}^{\infty}(EV_a - y_n^2)\frac{Kny_n}{\sin y_n}\exp(-y_n^2\tau) \quad (16)$$

fluid:

$$V_f = V_\sigma + \frac{1}{Bi}\frac{\partial V_s}{\partial x}\bigg|_{1,\tau} = V_a - 2\sum_{n=1}^{\infty}(EV_a - y_n^2)Kn \times \frac{(1+y_n\cos y_n - \sin y_n)}{Bi\sin y_n}\exp(-y_n^2\tau). \quad (19)$$

4. ENERGY BALANCE SOLUTION

We now derive separately the solution for the special case $k \to \infty$.

In this case $T_s \equiv T_\sigma \equiv \hat{T}_s \equiv T_c$ and we may write the governing equations as

$$M_s C_s \frac{dT_s}{dt} = -4\pi R^2 h(T_s - T_f) \tag{20}$$

$$M_f C_f \frac{dT_f}{dt} = M_s C_s \frac{dT_s}{dt} + UA(T_f - T_a) \tag{21}$$

with $T_s(0) = T_{s0}$, $T_f(0) = T_{f0}$ as before.

Define:

$$E^* = UAH/4\pi R^2 h \tag{22}$$

and

$$\tau^* = 4\pi R^2 ht/M_s C_s \tag{23}$$

to obtain the following set of dimensionless equations

$$\frac{dV_s}{d\tau^*} = V_f - V_s \tag{24}$$

$$-\frac{dV_f}{d\tau^*} = H\frac{dV_s}{d\tau^*} + E^*(V_f - V_a) \tag{25}$$

$$V_s(0) = 0 \tag{26}$$

$$V_f(0) = 1. \tag{27}$$

These equations are essentially those given by Kern [8] in the analysis of a heat exchanger with ambient loss.

The solution is

$$V_{se} = V_a + \left[\frac{2-(\beta+\Delta)V_a}{2\Delta}\right]\exp\left[\frac{(\Delta-\beta)\tau^*}{2}\right] - \left[\frac{2-(\beta-\Delta)V_a}{2\Delta}\right]\exp\left[-\frac{(\Delta+\beta)\tau^*}{2}\right] \tag{28}$$

$$V_{fe} = V_a + \left\{\frac{[2-(\beta+\Delta)V_a][2+\Delta-\beta]}{4\Delta}\right\} \times \exp\left[\frac{(\Delta-\beta)\tau^*}{2}\right] - \left\{\frac{[2-(\beta-\Delta)V_a][2-\Delta-\beta]}{4\Delta}\right\} \times \exp\left[-\frac{(\Delta+\beta)\tau^*}{2\Delta}\right] \tag{29}$$

where $\beta = H + E^* - 1$
$\Delta = (\beta^2 - 4E^*)^{1/2}$.

When Bi is very small, but not zero, the true solution (13) is approximated by the energy balance solution (28) with E^* and τ^* the following functions of E, Bi, τ:

$$E^* = E/3Bi \tag{30}$$

$$\tau^* = 3Bi\tau. \tag{31}$$

We recall that for simpler versions of the problem $(H = E = 0)Bi < 0.1$ is an accepted criterion for applying the energy balance solution. We see from Fig. 2 that for the general problem such a criterion is safe for $\tau > 0.01$ and possibly for smaller τ as well.

Note that the error in the energy balance solution now has two components, ε_1 and ε_2:

true solution:

$$\hat{V}_s = f(\tau) + V_a g(\tau) \tag{32}$$

energy balance solution:

$$V_{se} = f(\tau)[1+\varepsilon_1(\tau)] + V_a g(\tau)[1+\varepsilon_2(\tau)] \tag{33}$$

fractional error:

$$Er = \frac{f\varepsilon_1 + V_a \varepsilon_2 g}{f + V_a g}$$

and hence for any fixed τ

$$Er \begin{array}{l} \to \varepsilon_1 \text{ as } V_a \to 0 \\ \to \varepsilon_2 \text{ as } V_a \to \infty \end{array}$$

Er has no proper maximum, so the larger of ε_1 and ε_2 is the maximum.

In Fig. 2 $f(\tau)$, $g(\tau)$, $\varepsilon_1(\tau)$, $\varepsilon_2(\tau)$ are plotted against τ for $0.01 < \tau < 10$ for the case $Bi = 0.1$, $H = 1$, $E = 1$. The functions analogous to f and g in the fluid, surface and centre temperatures (f_a, f_c, f_f; g_a, g_c, g_f) are also plotted, the functions analogous to ε_1 and ε_2 were too small to be plotted for the fluid temperature. Note that on the graph the centre, average and surface temperatures coincide.

From Fig. 2 $|Er| < 3\%$.

The energy balance solution is clearly much easier to apply than the exact solution since we do not need to calculate the roots of a transcendental equation. In addition the full solution may become very sensitive to the accuracy of the first root of the transcendental equation. However, this phenomenon occurs predominantly at very small Biot numbers where the energy balance solution is to be preferred anyway.

5. LIMITING SOLUTIONS FOR HIGH AND LOW PARAMETER VALUES

For cases where one of the parameters Bi, H, E has a value that is much larger or smaller than the others, the limiting solutions given below will provide useful approximations and are considerably easier to use than the exact result. These solutions fall into two classes, those that may be obtained by simplifying the general solution (13) and those which cannot be obtained in this way because some of the equations (7)–(12) become meaningless.

(1) $Bi \to \infty$ and H, E stay finite; the transcendental equation (15) reduces to

$$\frac{\tan y}{y} = \frac{3H}{y^2 - E + 3H} \tag{34}$$

and the coefficients Kn reduce to

$$Kn = \frac{3H}{y_n^4 + [9H(1+H) - 2E]y_n^2 + E^2 - 3EH}. \tag{35}$$

Note that with $E \to 0$ our solution reduces to Paterson's [3].

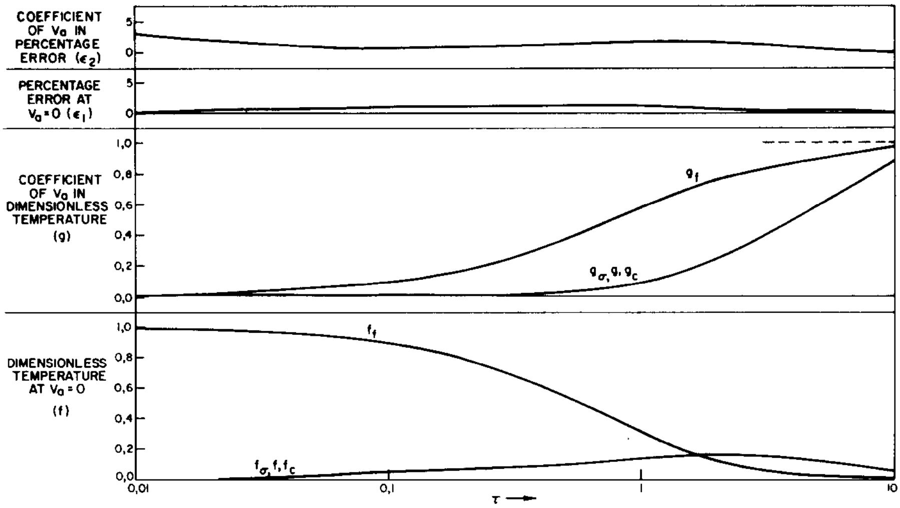

FIG. 2. Dimensionless temperature–time profiles for low Biot number ($Bi = 0.1$; $H = 1$; $C = 1$).

(2) $Bi \to 0$ and H, E stay finite; we have

$$\frac{\tan y}{y} = 1 \tag{36}$$

and

$$Kn = 0. \tag{37}$$

This implies $V_s = V_a$, so the result from equations (13)–(15) is not valid. This happens because $Bi = 0$ implies $\partial V_s/\partial x = 0$ and equation (7) becomes $0 = 0$, so (7) and (8) fall away and are replaced by (20), (11) is no longer necessary and the remaining equations (9), (10), (12) together with (20) lead to the energy balance solution (28).

(3) $H \to 0$; E, Bi non-zero. Physically this is the case of a very large fluid extent losing heat to ambient, so the fluid temperature should depend only on E. $H = 0$ implies there is no solid, so Bi is not defined, and equations (7), (8), (10), (11) become meaningless. (9) reduces to

$$-\frac{dV_f}{d\tau\dagger} = (V_f - V_a) \tag{38}$$

$$V_f = V_a + (V_{f0} - V_a)\exp(-\tau\dagger) \tag{39}$$

where

$$\tau\dagger = E\tau = \frac{UAt}{M_f C_f}.$$

(4) $H \to \infty$; E, Bi finite. This implies $M_s C_s \to \infty$ (if $M_f C_f \to 0$, $E \to \infty$) and is not a well posed problem.

(5) $E \to 0$; H, Bi finite. This reduces to the case treated in our earlier paper [2]. Note that in the present formulation V_s is defined relative to the ambient temperature, but the ambient is no longer involved in the process. However, for $E \neq 0$, T_a can be interpreted as the equilibrium temperature of the system. So redefine T_a for $E = 0$ to be the equilibrium temperature

$$T_a = \frac{M_f C_f T_{f0} + M_s C_s T_{s0}}{M_f C_f + M_s C_s}$$

and regenerate our earlier result [2], which in the present nomenclature is:

$$\frac{\tan y}{y} = \frac{3BiH - y^2}{(Bi-1)y^2 + 3BiH} \tag{40}$$

$$Kn(EV_a - y_n^2) = \frac{Bi(3BiH - y_n^2)}{y_n^4 + [Bi(Bi-1) - 6BiH]y_n^2 + 9Bi^2H(1+H)}. \tag{41}$$

(6) $E \to \infty$; H, Bi finite.

$$\frac{\tan y}{y} = \frac{-1}{(Bi-1)} \tag{42}$$

$$Kn(EV_a - y_n^2) = \frac{BiV_a}{y_n^2 + Bi(Bi-1)}. \tag{43}$$

As expected, H is no longer a parameter since $E = \infty$ implies that the ambient is a continuation of the fluid [since (9) implies $V_f = V_a$] so we have heat transfer to an infinite fluid at V_a.

The above solution may be compared to Carslaw and Jaeger's [9] solution for heat transfer to an infinite fluid, which in our nomenclature is:

$$\frac{V_s}{V_a} = \frac{2}{x}\sum \exp(-y_n^2\tau)\frac{y_n^2 + (Bi-1)^2}{y_n^2 + Bi(Bi-1)} \times \sin(y_n x)\int_0^1 x\sin(y_n x)\,dx. \tag{44}$$

To show that the solutions are equivalent we must show that

$$[y_n^2 + (Bi-1)^2]\int_0^1 x\sin(y_n x)\,dx = \frac{Bi}{\sin y_n};$$

Left hand side:

$$\text{LHS} = [y_n^2 + (Bi-1)^2]\frac{(\sin y_n - y_n\cos y_n)}{y_n^2}. \tag{45}$$

Square equation (42), invert and add y_n^2 to each side to get

$$y^2(1 + \cot^2 y) = y^2 + (Bi-1)^2 \tag{46}$$

use $1/\sin^2 y = 1+\cot^2 y$ in (46), and (46) in (45) to get

$$\mathrm{LHS} = \frac{1-y_n \cot y_n}{\sin y_n} = \frac{Bi}{\sin y_n}.$$

So equation (45) is an identity and therefore the solutions are the same.

6. APPLICATION TO CONTINUOUS PROCESSES

There are several systems of industrial importance that obey the same equations as the non-adiabatic calorimeter.

(1) *Co-current moving bed contactor* (*see Fig.* 3)

If the fluid and solid streams are both in plug flow then the time that the solids have spent in the contactor when they are at position l is given by

$$t = \frac{l\rho_s S}{G_s}. \tag{47}$$

Equations (1), (2), (4)–(6) apply.

The overall heat balance on the element of contactor between l and $l+\mathrm{d}l$ is

$$-G_f C_f \,\mathrm{d}T_f = G_s C_s \,\mathrm{d}\hat{T}_s + U(l)A\frac{\mathrm{d}l}{L}(T_f - T_a). \tag{48}$$

Under many circumstances the assumption that $U(l) = U$ = constant is approximately true and in that case equations (1), (2), (48), (4)–(6) lead to the set of equations (7)–(12). Where V_s, V_f, V_a, x, Bi have the same definition as previously and τ, H and E are given by

$$\tau = \frac{\alpha\rho_s S l}{G_s R^2} \tag{49}$$

$$H = \frac{G_s C_s}{G_f C_f} \tag{50}$$

$$E = \frac{UAR^2 G_s}{\alpha G_f C_f \rho_s S}. \tag{51}$$

In contrast with the calorimeter where $0 < \tau < \infty$ we now have

$$0 < \tau < \frac{\alpha\rho_s SL}{G_s R^2}.$$

This does not affect the solution since we can increase the length of the column from L to L' without altering the solid or fluid temperature fields in the section of column $0 < l < L$. This carries the implication that it is possible to find the length required for a given duty without iteration.

As is the case with heat exchangers with ambient loss [8], it is not always possible to design a contactor for a given duty when the value of U is specified.

For instance, if we are attempting to heat up a solid using a hot fluid, heat loss to the ambient may be so large that the solid can never reach the desired temperature. This is because the solid average temperature reaches a maximum whose position and value are functions of the magnitude of heat loss to the ambient. One application of the above model would be the optimisation of the insulation thickness on a high temperature contactor.

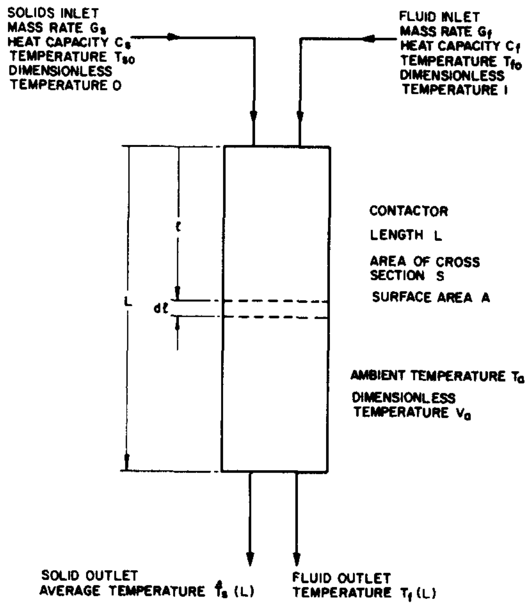

FIG. 3. Co-current moving bed contactor.

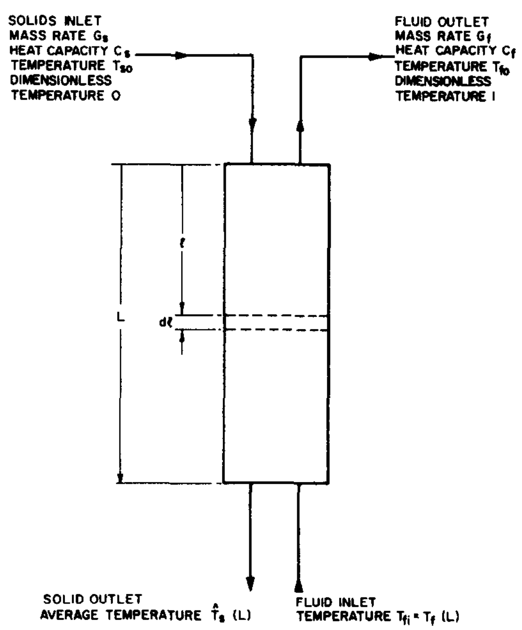

FIG. 4. Counter-current moving bed contactor.

(2) *Counter-current moving bed contactor* (*see Fig.* 4)

The governing equations are almost identical to those of the co-current case, differing only in the heat balance equation which becomes

$$G_f C_f \,\mathrm{d}T_f = G_s C_s \,\mathrm{d}\hat{T}_s + U(l)A\frac{\mathrm{d}l}{L}(T_f - T_a). \tag{52}$$

So in the present case we have to redefine E and H, i.e.

$$E = -\frac{UAR^2G_s}{\alpha G_f C_f \rho_s S} \quad (53)$$

$$H = -\frac{G_s C_s}{G_f C_f}. \quad (54)$$

Note that E and H are now always negative. And equations (7)–(12) can be generated by defining the other parameters and variables in exactly the same way as in the co-current case.

It is seen that the problem has been transformed into a hypothetical non-adiabatic calorimeter containing a fluid of negative heat capacity, and that we have specified the outlet rather than the inlet fluid temperature as T_{f0}. Without this modification a change in length of the contactor, leaving T_{fi} fixed, would change the temperature profile throughout the contactor and the domain of τ would no longer be equivalent to a semi-infinite domain. Although in practice we would not always know the fluid outlet temperature to start with, it is easily determined as follows:

$$\frac{T_{fi}-T_{s0}}{T_{f0}-T_{s0}} = V_f|_{l=L}. \quad (55)$$

The solution V_f is known in the form

$$V_f = f(\tau) + \frac{T_a - T_{s0}}{T_{f0}-T_{s0}} g(\tau) \quad (56)$$

hence

$$T_{f0} - T_{s0} = \frac{(T_{fi}-T_{s0})}{f|l=L} + (T_a - T_{s0})\frac{g|l=L}{f|l=L}. \quad (57)$$

Thus $T_{f0}-T_{s0}$ may be calculated directly given L and the other parameters and hence the other dimensionless temperatures easily converted to actual temperatures.

On the other hand, this means that calculation of column length required for a given outlet solids average temperature requires iteration on L, unlike the co-current case where L may be calculated directly.

The remarks about feasibility and optimisation of insulation thickness made for the co-current case still apply.

(3) *Cascade of fluidised beds* (*see Fig.* 5)

If there are sufficiently many stages, the fluidised bed cascade behaves approximately like a counter-current plug flow contactor and may be treated as such (see the previous case). With few stages, on the other hand, the analysis of Kasten and Amundson [10] may be used if heat loss to the ambient is small.

In passing through each fluidised bed of the cascade a particle spends some time in counter-current flow, some in cross flow and some in co-current flow. So the performance of such a cascade, with a dimensionless mean residence time $\bar{\tau}$, would be better than that of a co-current contactor of dimensionless length $\bar{\tau}$ and worse than that of a similar counter-current contactor (the Biot number applicable to the cascade, not that which would apply to a true plug flow contactor must of course be used). So the solution for the co-current contactor would lead to a conservative design for the cascade and the performance of the equivalent counter-current contactor would be a measure of how conservative the design is.

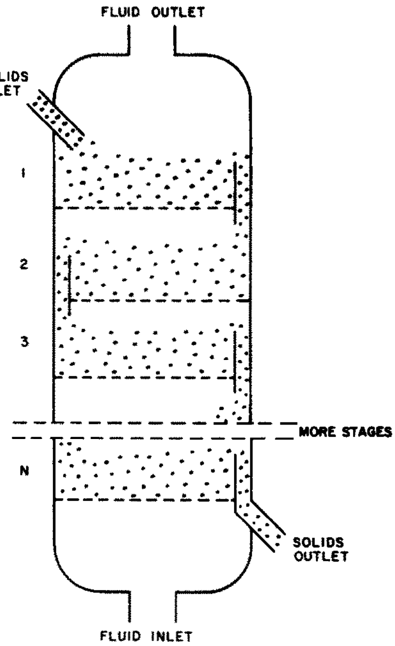

FIG. 5. Cascade of fluidised beds.

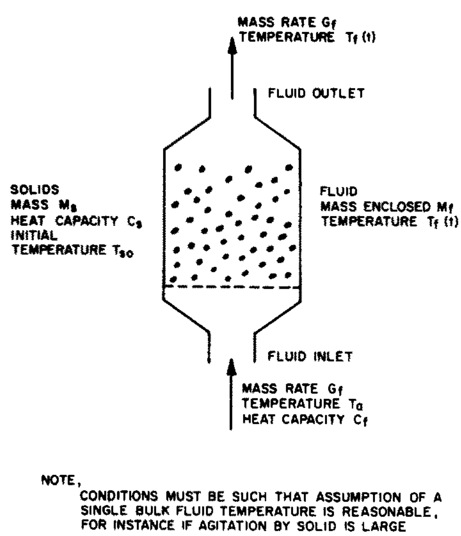

FIG. 6. Fluidised bed with a batch of solids and a stream of fluid.

(4) *Fluidised bed with a batch of solids and continuous flow of fluid* (*see Fig.* 6)

When it is reasonable to assume that the fluid has a bulk temperature $T_f(t)$, equations (1), (2), (4)–(6)

apply and the energy balance reads

$$-M_f C_f \mathrm{d}T_f = M_s C_s \mathrm{d}\hat{T}_s + G_f C_f (T_f - T_a)\mathrm{d}t. \quad (58)$$

We can make the system of equations (1), (2), (58), (4)–(6) dimensionless and generate equations (7)–(12) by defining Bi, H, x, τ, V_s, V_f, V_a as in the batch case and further

$$E = \frac{G_f R^2}{M_f \alpha}. \quad (59)$$

Here T_a is the inlet fluid temperature and E is the dimensionless reciprocal mean residence time of the fluid.

The above case has a fairly common mass transfer analogue, namely the drying of grain or peas, where moisture movement is diffusion controlled inside the solid and convection controlled at the surface. For such a system, the mass-transfer problem fits the above model and the heat-transfer problem is essentially that of transfer from the fluid through a film to a heat sink on the solid surface and is thus easily dealt with.

7. DISCUSSION OF THE ROOTS OF THE TRANSCENDENTAL EQUATION

In cases where we cannot use the energy balance solution, it is necessary to solve the transcendental equation:

$$\frac{\tan y}{y} = \frac{3BiH + E - y^2}{(Bi-1)(y^2 - E) + 3BiH}. \quad (15)$$

The series solution for the temperature contains terms in $\exp(-y_n^2\tau)$. In the batch case y_n must be real for all n, otherwise the temperature would not tend to V_a as time becomes infinite.

In the continuous-counter-current application this is no longer true and we expect exponential growth terms in the T vs l profile if the solids heat capacity rate is greater than that of the fluid so that the fluid undergoes most of its temperature change near its inlet, i.e. the solid outlet. So for $H < -1$ there are some purely imaginary roots whereas for $H > -1$ all the roots are real.

More general complex roots may always be ruled out since they contribute an oscillating component to the solution, which is physically impossible. Still, when solving equation (15) for its roots, one has to take care in finding all the roots that play a significant part in the solution. A qualitative check may be obtained from the answer at $\tau = 0$. If the latter is not satisfactory one may examine one of the following possible reasons: (a) not enough terms considered, i.e. truncation error; (b) roots not located accurately enough; (c) roots that are significant have been inadvertently omitted.

The first and second are easy to deal with, the third is not so easily dealt with. However, if there is reason to suspect that roots have been omitted, it is worthwhile to draw a graph of

$$\phi(y) = \frac{\tan y}{y} \text{ and } \psi(y) = \frac{3BiH + E - y^2}{3BiH + (Bi-1)(y^2 - E)}$$

against y such as Fig. 7 which is for $Bi = 0.5$, $H = 1$, $E = 1$.

Depending on the parameter values there may be one, two or even no roots within an interval from $(n-\frac{1}{2})\pi$ to $(n+\frac{1}{2})\pi$.

When $H < -1$ there are also imaginary roots of (15), i.e. roots of:

$$\frac{\tanh w}{w} = \frac{w^2 + E + 3HBi}{(1-Bi)(w^2+E) + 3HBi} \quad (60)$$

where $w = -iy$.

Again a graph may be useful.

8. CONCLUSIONS

We have solved the problem of a solid sphere in contact with a fixed mass of fluid, which exchanges heat simultaneously with the solid and with the ambient. The general solution for a solid of finite

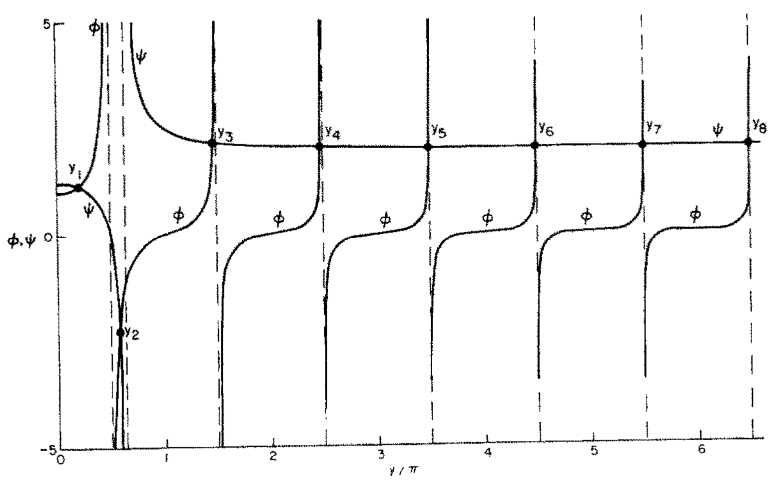

FIG. 7. Graph of $\phi(y) = \tan y/y$ and $\psi(y) = (3BiH + E - y^2)/[(Bi-1)(y^2 - E) + 3BiH]$ showing the first few roots of equation (15) for $Bi = 0.5$; $H = 1$; $E = 1$.

thermal conductivity is available as an infinite series with four parameters, Bi, H, E and V_a. Few terms of the series suffice for most applications because the terms contain a strong exponential decay, of the form $\exp(-y_n^2\tau)$.

Some of the problems which are encountered in solving the transcendental equation for the roots y_n may be overcome by applying appropriate checks or drawing an appropriate graph.

The results find application in the analysis of, and design of equipment for, continuous solid–fluid exchange processes. Various simplified solutions for limiting values of the parameters Bi, H, E, are presented. Among these is the energy balance solution for small Bi, which reduces the problem to that of a double-pipe heat exchanger with ambient loss.

9. REFERENCES

1. F. J. Edeskuty and N. R. Amundson, Mathematics of adsorption. IV. Effect of intraparticle diffusion in agitated static systems, *J. Phys. Chem.* **56**, 148–152 (1952).
2. J. Kern and J. W. Hemmings, On the analogy between the calorimeter problem and some granulate-fluid exchange processes, *J. Heat Transfer* **100**, 319–323 (1978).
3. S. Paterson, Heating or cooling of a solid sphere in a well-stirred fluid, *Proc. Phys. Soc.* **59**, 50–58 (1947).
4. M. S. Brinn, S. J. Friedman, F. A. Gluckert and R. L. Pigford, Heat transfer to granular materials—settled beds moving downward through vertical tubes, *Ind. Engng Chem.* **40**, 1050–1061 (1948).
5. P. R. Kasten and N. R. Amundson, Analytical solution for simple systems in moving bed adsorbers, *Ind. Engng Chem.* **44**, 1704–1711 (1952).
6. W. D. Munro and N. R. Amundson, Solid–fluid exchange in moving beds, *Ind. Engng Chem.* **42**, 1481–1488 (1950).
7. C. W. Siegmund, W. D. Munro and N. R. Amundson, Solid–fluid reactions in fixed and moving beds, *Ind. Engng Chem.* **48**, 43–50 (1956).
8. J. Kern, On the feasibility of heat-exchange processes in uninsulated equipment, *Wärme- und Stoffübertrung* **9**(4), 267–272 (1976).
9. H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids*, Paragraph 92. Clarendon Press, Oxford (1947).
10. P. R. Kasten and N. R. Amundson, An elementary theory of adsorption in fluidised beds, *Ind. Engng Chem.* **42**, 1341–1346 (1950).

APPENDIX

Solution of equations (7)–(12) *by Laplace transforms*

Transform equations (7)–(12).

$$sx\bar{V}_s = \frac{d^2}{dx^2}x\bar{V}_s \tag{61}$$

$$\left.\frac{d\bar{V}_s}{dx}\right|_{x=0} = 0 \tag{62}$$

$$\left.\frac{d\bar{V}_s}{dx}\right|_{x=1} = Bi[\bar{V}_f - \bar{V}_s(1,s)] \tag{63}$$

$$-s\bar{V}_f + 1 = Hs\bar{V}_s + E\left(\bar{V}_f - \frac{V_a}{s}\right) \tag{64}$$

$$\bar{\hat{V}}_s = \hat{\bar{V}}_s = 3\int_0^1 x^2\bar{V}_s\,dx. \tag{65}$$

General solution of (61) is:

$$x\bar{V} = A\cosh[(s)^{1/2}x] + B\sinh[(s)^{1/2}x].$$

(62) leads to $A \equiv 0$

let $w = (s)^{1/2}$.

Then

$$\bar{V}_s = \frac{B(w)\sinh(wx)}{x}. \tag{66}$$

(65) leads to

$$\hat{\bar{V}}_s = 3B(w)(w\cosh w - \sinh w)/w^2. \tag{67}$$

(67) and (64) lead to

$$\bar{V}_f = \frac{1 + \frac{EV_a}{w^2} - 3HB(w)(w\cosh w - \sinh w)}{w^2 + E}. \tag{68}$$

Apply (66)–(68) in (63) to obtain

$$B = \frac{Bi(1 + EV_a/w^2)}{(w^2 + E + 3HBi)w\cosh w - [(1 - Bi)(w^2 + E) + 3HBi]\sinh w}. \tag{69}$$

Now use Cauchy's integral theorem on the Mellin integral to invert the transform

$$\mathscr{L}^{-1}[F(s)] = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{s\tau}F(s)\,ds = \sum_{\forall_i} \operatorname{Res}_{s=s_i}[e^{s\tau}F(s)]$$

where s_i are the isolated singularities of F on the complex plane, and γ is an arbitrary number such that real $\{s_i\} > \gamma$ for all i.

s_i will be the poles of $B(w)$, i.e. $s_i = w_i^2$ where w_i are the roots of:

$$\frac{\tanh w}{w} = \frac{w^2+E+3HBi}{(1-Bi)(w^2+E)+3HBi}. \tag{70}$$

Note that $w_0 = 0$ is also a pole of B, but has been excluded from equation (70) by performing algebra on the denominator of B that is invalid if $w = 0$.

Treat w_0 separately because $\dfrac{\sinh(wx)}{x}$ is also zero here.

Use $\operatorname*{Res}_{s=s_i} F(s) = \lim_{s\to s_i}(s-s_i)F(s)$, which is the definition of the residue at a simple pole, for $s_0 = w_0^2 = 0$. We get

$$\begin{aligned}\operatorname*{Res}_{s=0}[e^{s\tau}F(s)] &= \lim_{w\to 0}\left[w^2B(w)\frac{\sinh wx}{x}\exp(w^2\tau)\right]\\ &= \lim_{w\to 0} w^3B(w)\\ &= V_a.\end{aligned}$$

For the other singularities, which are all simple poles, use

$$\operatorname*{Res}_{s=s_i}\frac{F(s,x,\tau)}{G(s)} = \lim_{s\to s_i}\frac{F(s,x,\tau)}{\dfrac{dG}{ds}} = \lim_{w\to w_i}\frac{2wF(w^2,x,\tau)}{\dfrac{dG(w^2)}{dw}}$$

So

$$\operatorname*{Res}_{s=s_i}\frac{\sinh(s)^{1/2}x}{x}e^{s\tau}B(s)^{1/2} = \frac{\sinh w_i x}{x}e^{w_i^2\tau}\operatorname*{Res}_{s=w_i^2}B(s)^{1/2}.$$

After some differentiation we get

$$\operatorname*{Res}_{s=s_i}B(s)^{1/2} = \lim_{w\to w_i}\frac{2wBi(1+EV_a/w^2)}{[w^2+E+3HBi-2(1-Bi)]w\sinh w(3w^2+E-[1-Bi][w^2+E])\cosh w}.$$

From (68) we get an expression for $\cosh w$ in terms of w and $\sinh w$

$$\cosh w_i = \frac{(1-Bi)(w_i^2+E)+3HBi}{w_i(w_i^2+E+3HBi)}\sinh w_i.$$

$$\operatorname*{Res}_{s=s_i}B(s)^{1/2} = \frac{2w_i^2(w_i^2+E+3HBi)Bi(1+EV_a/w_i^2)}{\sinh w_i\{[w_i^2+E+3HBi-2(1-Bi)]w_i^2(w_i^2+E+3HBi)+[3w_i^2+E-(1-Bi)(w_i^2+E)][(1+Bi)(w_i^2+E)+3HBi]\}}$$

Now, since all the solutions, w_i, are purely imaginary (in the batch case), we substitute $y_i = (-w_i^2)^{1/2} = iw_i$ in the equations, purely for convenience.

In this manner we obtain equations (13)–(15).

LE PROBLEME DU CALORIMETRE NON ADIABATIQUE ET SON APPLICATION AUX MECANISMES DE TRANSFERT DANS LES SUSPENSIONS SOLIDES

Résumé – On résout le problème du transfert simultané de chaleur d'un fluide d'étendue finie vers l'ambiance et vers un solide de conductivité thermique finie. La solution générale est sous forme d'une série infinie en fonction du nombre de Biot, du rapport H d'étendue thermique, du paramètre E de perte ambiante et de la température ambiante V_a; la série montre une forte décroissance exponentielle et il suffit de peu de termes pour la plupart des applications. La solution est une extension et une amélioration de celle du problème classique du calorimètre. Les résultats trouvent une application dans l'analyse (et dans le dimensionnement du matériel) de l'échange continu avec des solides en suspension. La forme de la solution est suffisamment simple pour permettre le calcul d'optimisation de l'épaisseur d'isolant, du temps de résidence et d'autres paramètres. On présente des solutions simples pour les valeurs limites des paramètres, par exemple, pour le tube annulaire de l'échangeur de chaleur avec perte à l'ambiance.

DAS VERALLGEMEINERTE KALORIMETERPROBLEM UND SEINE ÜBERTRAGUNG AUF AUSTAUSCHPROZESSE IN FLUID-FESTSTOFF-GEMISCHEN

Zusammenfassung – Es wird der gleichzeitige Wärme- oder Stoffaustausch zwischen einem endlich ausgedehnten Fluid, einem endlich ausgedehnten Feststoff von endlicher Leitfähigkeit und einer unendlich ausgedehnten Umgebungsatmosphäre analytisch behandelt. Die Lösung des derart verallgemeinerten Kalorimeterproblems lässt sich verschiedenartig interpretieren, so dass eine Reihe wichtiger Austauschprozesse in Gas-Feststoff-Systemen damit ausgelegt werden können. Wegen der notwendigen Vereinfachung des Problems sind die zu erwartenden Ergebnisse zwar nicht immer numerisch exakt, andererseits ermöglicht die einfache Form der Lösung weiterführende Untersuchungen, etwa zum Zwecke der Prozessoptimierung.

ЗАДАЧА НЕАДИАБАТИЧЕСКОГО КАЛОРИМЕТРА И ЕЁ ПРИМЕНЕНИЕ К ПРОЦЕССАМ ПЕРЕНОСА В СУСПЕНЗИЯХ ТВЕРДЫХ ЧАСТИЦ

Аннотация — Представлено решение задачи одновременного переноса тепла от конечного объёма жидкости в окружающую среду и к твердому телу с конечной теплопроводностью. Общее решение даётся в виде бесконечного ряда, в который в качестве безразмерных параметров включены: число Био, параметр теплового расширения H, параметр потерь в окружающую среду E и температура окружающей среды V_a. Данный ряд имеет экспоненциальную сходимость, так что для практических целей достаточно нескольких членов ряда. Решение представляет собой обобщение и модификацию классической задачи калориметра. Результаты могут использоваться для анализа процесса непрерывного теплообмена со взвешенными твердыми частицами и для расчёта соответствующей аппаратуры. Решение имеет довольно простой вид и может использоваться для расчёта оптимальной толщины изоляции, времени контакта частиц и других параметров процесса. Для предельных значений параметров приведены упрощенные решения, как например, для теплообменника типа «труба в трубе» при наличии потерь тепла в окружающую среду.